

Borel Liftings of Graph Limits

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Abstract

The cut pseudo-metric on the space of graph limits induces an equivalence relation. The quotient space obtained by collapsing each equivalence class to a point is a metric space with appealing analytic properties. We show that the equivalence relation admits a Borel lifting: There exists a Borel-measurable mapping which maps each equivalence class to one of its elements.

Let $(\Omega, \mathcal{B}(\Omega), P)$ be an atomless Borel probability space and $L_1(\Omega^2)$ the Banach space of integrable functions on $\Omega \times \Omega$, equipped with the L_1 -metric d_1 . Let $\mathbf{W} \subset L_1(\Omega^2)$ be the subspace of symmetric integrable functions $\Omega^2 \rightarrow [0, 1]$. Define a pseudo-norm on \mathbf{W} by

$$\|w\|_{\square} = \sup_{S, T \in \mathcal{B}(\Omega)} \int_{S \times T} w(s, t) dP(s) dP(t) . \quad (1)$$

Following [1], we use $\|\cdot\|_{\square}$ to define a pseudo-metric on \mathbf{W} as

$$\delta_{\square}(w, w') := \inf_{\psi} \|w^{\psi} - w'\|_{\square} \quad \text{where} \quad w^{\psi}(x, y) = w(\psi(x), \psi(y)) . \quad (2)$$

The infimum is taken over all invertible measure-preserving transformations of Ω , i.e. all invertible measurable mappings $\psi : \Omega \rightarrow \Omega$ satisfying $\psi P = P$. The pseudo-metric induces an equivalence relation on \mathbf{W} , given by $w \equiv w' \Leftrightarrow \delta_{\square}(w, w') = 0$. The relation $w \equiv w'$ is also known as *weak isomorphy* of w and w' [6]. Denote the equivalence class of $w \in \mathbf{W}$ by $[w]_{\square}$, and the quotient space of all equivalence classes by $\widehat{\mathbf{W}}$. On the quotient space, δ_{\square} is a metric, and the metric space $(\mathbf{W}, \delta_{\square})$ is known to be compact [8]. For each $\widehat{w} \in \widehat{\mathbf{W}}$, we write $[\widehat{w}]_{\square} \subset \mathbf{W}$ for the corresponding equivalence class of elements of \mathbf{W} .

Theorem 1 below shows that weak isomorphy admits a Borel lifting, i.e. there exists a Borel-measurable mapping $\xi : (\widehat{\mathbf{W}}, \delta_{\square}) \rightarrow (\mathbf{W}, d_1)$ such that

$$\xi(\widehat{w}) \in [\widehat{w}]_{\square} \quad \text{for all } \widehat{w} \in \widehat{\mathbf{W}} . \quad (3)$$

The lifting is not unique. More precisely:

Theorem 1. *There is a sequence (ξ_n) of measurable mappings $\xi_n : (\widehat{\mathbf{W}}, \delta_{\square}) \rightarrow (\mathbf{W}, d_1)$ such that, for every $\widehat{w} \in \widehat{\mathbf{W}}$, the set $\{\xi_n(\widehat{w}) \mid n \in \mathbb{N}\}$ is a dense subset of $[\widehat{w}]_{\square}$.*

Proof. Theorem 1 can be stated equivalently by defining a set-valued mapping

$$\phi_{\square} : \widehat{\mathbf{W}} \rightarrow 2^{\mathbf{W}} \quad \text{with} \quad \phi_{\square}(\widehat{w}) := [\widehat{w}]_{\square}. \quad (4)$$

We then have to show that there are measurable mappings ξ_n with

$$\overline{\{\xi_n(\widehat{w}) \mid n \in \mathbb{N}\}} = \phi_{\square}(\widehat{w}) \quad \text{for all } \widehat{w} \in \widehat{\mathbf{W}}, \quad (5)$$

where \overline{A} denotes the closure of A .

Liftings of set-valued maps are a well-studied topic in analysis, and we use a result of Kuratowski and Ryll-Nardzewski [5] on the existence of liftings, and a generalization by Castaing [2] (see e.g. [3, Theorem 12.16] and [4, Theorem 14.4.1] for textbook statements). For our purposes, these results can be summarized as follows:

Theorem 2. *Let \mathbf{X} be a measurable space, \mathbf{Y} a Polish space, and $\phi : \mathbf{X} \rightarrow 2^{\mathbf{Y}}$ a set-valued mapping. Require $\phi(x)$ to be non-empty and closed for all $x \in \mathbf{X}$, and that*

$$\phi^{-1}(A) := \{x \in \mathbf{X} \mid \phi(x) \cap A \neq \emptyset\} \quad (6)$$

is a measurable set in \mathbf{X} for each open set A in \mathbf{Y} . Then there exists a sequence of measurable mappings $\xi_n : \mathbf{X} \rightarrow \mathbf{Y}$ such that $\overline{\{\xi_n(x) \mid n \in \mathbb{N}\}} = \phi(x)$ for all $x \in \mathbf{X}$.

For ϕ_{\square} as defined in (4) and any subset $A \subset \mathbf{W}$, the set $\phi_{\square}^{-1}(A)$ in (6) simply consists of all $\widehat{w} \in \widehat{\mathbf{W}}$ for which A contains at least one element of the equivalence class $[\widehat{w}]_{\square}$. If A is in particular an open d_1 -ball in \mathbf{W} , this set has the following property:

Lemma 3. *Denote by $U_{\varepsilon}(v)$ the open d_1 -ball of radius ε centered at $v \in \mathbf{W}$. If $\varepsilon < \delta$,*

$$\overline{\phi_{\square}^{-1}(U_{\varepsilon}(v))} \subseteq \phi_{\square}^{-1}(U_{\delta}(v)) \quad (7)$$

for all $v \in \mathbf{W}$.

Proof of Lemma 3. Let $(\widehat{w}_1, \widehat{w}_2, \dots)$ be a sequence in $\phi_{\square}^{-1}(U_{\varepsilon}(v))$ with $\widehat{w}_i \xrightarrow{\delta_{\square}} \widehat{w}$. We have to show that $\widehat{w} \in \phi_{\square}^{-1}(U_{\delta}(v))$. By definition of ϕ_{\square}^{-1} , the sets $\phi_{\square}(\widehat{w}_i) \cap U_{\varepsilon}(v)$ are non-empty. Suppose (w_i) is a sequence with $w_i \in \phi_{\square}(\widehat{w}_i) \cap U_{\varepsilon}(v)$ for each $i \in \mathbb{N}$. By [9, Lemma 2.11], convergence of (\widehat{w}_i) to \widehat{w} then implies

$$\varepsilon \geq \liminf \delta_1(w_i, v) \geq \delta_1(w, v) = \inf_{\psi} d_1(w^{\psi}, v) \quad (8)$$

for any $w \in \phi_{\square}(\widehat{w})$. Since $\varepsilon < \delta$, there is hence a measure-preserving transformation ψ such that $d_1(w^{\psi}, v) < \delta$, that is, $w^{\psi} \in U_{\delta}(v)$. Because w^{ψ} and w are weakly isomorphic, we also have $w^{\psi} \in \phi_{\square}(\widehat{w})$, and therefore

$$\widehat{w} \in \phi_{\square}^{-1}(\phi_{\square}(\widehat{w}) \cap U_{\delta}(v)) \subset \phi_{\square}^{-1}(U_{\delta}(v)). \quad (9)$$

□

Proof of Theorem 1. The space (W, d_1) is a closed subspace of the separable Banach space $L_1(\Omega^2)$ and hence Polish. The sets $\phi_{\square}(\widehat{w})$ are non-empty, by definition of the space $\widehat{\mathbf{W}}$ as a quotient. We will show that, additionally:

- i. The sets $\phi_{\square}(\widehat{w})$ are closed.
- ii. For all open sets A in \mathbf{W} , the set $\phi_{\square}^{-1}(A)$ is Borel in $\widehat{\mathbf{W}}$.

The mapping ϕ_{\square} therefore satisfies the hypothesis of Theorem 2, and Theorem 1 follows.

(i) Denote by $t_F: \mathbf{W} \rightarrow [0, 1]$ the homomorphism density indexed by a finite graph F [7]. Two elements of \mathbf{W} are weakly isomorphic if and only if their homomorphism densities coincide for all finite graphs F . Let $\widehat{w} \in \widehat{\mathbf{W}}$, and let (w_1, w_2, \dots) be a sequence in the set $\phi(\widehat{w})$ with limit w in (\mathbf{W}, d_1) . The homomorphism densities are δ_{\square} -continuous and hence d_1 -continuous. Therefore,

$$\lim t_F(w_i) = t_F(w) \quad \text{for all } F, \quad (10)$$

and since the w_i are weakly isomorphic, $t_F(w_i) = t_F(w)$ for all i and all F . Thus, $w \in \phi(\widehat{w})$, and the set is closed.

(ii) Let $U_{\delta}(v)$ denote the open ball of radius δ centered at $v \in \mathbf{W}$. Since \mathbf{W} is Polish, the open balls form a base of the topology and it is sufficient to consider sets of the form $A = U_{\delta}(v)$. Let $\delta_i \in \mathbb{R}_+$ be an increasing sequence $\delta_i \rightarrow \delta$. Then, by Lemma 3,

$$\phi_{\square}^{-1}(U_{\delta}(v)) = \bigcup_i \phi_{\square}^{-1}(U_{\delta_i}(v)) \subseteq \bigcup_i \overline{\phi_{\square}^{-1}(U_{\delta_i}(v))} \stackrel{(7)}{\subseteq} \bigcup_i \phi_{\square}^{-1}(U_{\delta}(v)) = \phi_{\square}^{-1}(U_{\delta}(v)).$$

In particular, $\phi_{\square}^{-1}(U_{\delta}(v))$ is a countable union of the closed sets $\overline{\phi_{\square}^{-1}(U_{\delta_i}(v))}$, and hence Borel. \square

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